Europ. J. Combinatorics (1996) **17,** 629–636

Distance-regular Subgraphs in a Distance-regular Graph, III

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Let Γ be a distance-regular graph with $l(1, a_1, b_1) = 1$ and $c_{s+1} = 1$ for some positive integer s. We show the existence of a certain distance-regular graph of diameter s, containing given two vertices at distance s, as a subgraph in Γ.

1. INTRODUCTION

All graphs considered in this paper are undirected finite graphs without loops or multiple edges. Let Γ be a connected graph. We identify Γ with the set of its vertices.

For two vertices u and v in Γ, we denote by $\partial_\Gamma(u, v)$ the distance between u and v in Γ, i.e. the length of a shortest path connecting u and v in Γ. Let

$$\Gamma_j(u) = \{x \in \Gamma \mid \partial_\Gamma(u, x) = j\},$$
$$k_\Gamma(u) = |\Gamma_1(u)|$$

and

$$d_\Gamma(u) = \max\{\partial_\Gamma(u, x) \mid x \in \Gamma\}.$$

For two vertices u and x in Γ with $\partial_\Gamma(u, x) = j$, let

$$C_j(u, x) = \Gamma_{j-1}(u) \cap \Gamma_1(x),$$
$$A_j(u, x) = \Gamma_j(u) \cap \Gamma_1(x)$$

and

$$B_j(u, x) = \Gamma_{j+1}(u) \cap \Gamma_1(x).$$

Γ is said to be *distance-regular* if

$$c_j(\Gamma) = |C_j(u, x)|, \qquad a_j(\Gamma) = |A_j(u, x)| \qquad \text{and} \qquad b_j(\Gamma) = |B_j(u, x)|$$

depend only on $j = \partial_\Gamma(u, x)$ rather than on individual vertices. It is easy to see that if Γ is a distance-regular graph, then $k_\Gamma(u)$ and $d_\Gamma(u)$ do not depend on the choice of u. Hence we write k_Γ and d_Γ. They are called the *valency* and the *diameter* of Γ. Sometimes we omit the suffix when the graph concerned is clear.

The numbers c_i, a_i and b_i are called the *intersection numbers* of Γ. The following are well known basic properties which we use implicitly in this paper.

Let u, v be adjacent vertices and let $x \in \Gamma_i(u) \cap \Gamma_{i+1}(v)$. Then:

(1) $C_i(u, x) \cup A_i(u, x) \cup B_i(u, x) = \Gamma_1(x)$. In particular, $c_i + a_i = b_i = k$.

(2) $C_i(u, x) \subset C_{i+1}(v, x)$ and $c_i \leqslant c_{i+1}$. Moreover, $C_i(u, x) = C_{i+1}(v, x)$ iff $c_i = c_{i+1}$.

(3) $B_i(u, x) \supset B_{i+1}(v, x)$ and $b_i \geqslant b_{i+1}$. Moreover, $B_i(u, x) = B_{i+1}(v, x)$ iff $b_i = b_{i+1}$.

The reader is referred to [1, 2] for the general theory of distance-regular graphs.

We first define the following notation and terminology used in this paper. Let

$$l(c, a, b) = \#\{j \mid (c_j, a_j, b_j) = (c, a, b)\}.$$

Let $x, y \in \Gamma$. We write $x \sim y$ when they are adjacent, and $x \not\sim y$ otherwise. When

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$\partial_\Gamma(x, y) = t$ and $c_t = 1$, we denote by $p[x, y]$ the unique shortest path connecting x and y.

Let X and Y be sets of vertices. We denote by

$$X - Y = \{x \in X \mid x \notin Y\}.$$

A quintuple of mutually distinct vertices $(x_0, x_1, x_2, x_3, x_4)$ is called a *pentagon* if $x_i \sim x_{i+1}$ and $x_i \not\sim x_{i+2}$ for any i, where indices are given by modulo 5.

In this paper, we prove the following theorem.

THEOREM 1.1. *Let Γ be a distance-regular graph with $l(1, a_1, b_1) = 1$. Assume that $c_{s+1} = 1$ for some positive integer s. For any two vertices $u, v \in \Gamma$ with $\partial_\Gamma(u, v) = s$, there exists a distance-regular graph $\Delta = \Delta(u, v)$ of diameter s, containing u and v as a subgraph in Γ. In particular, $k_\Delta = 1 + a_s(\Gamma)$, $c_j(\Delta) = c_j(\Gamma) = 1$ and $a_j(\Delta) = a_j(\Gamma)$ for $1 \leqslant j \leqslant s$.*

For the case $s = 1$, the result is trivial and well known. The result for $s = 2$ was proved by A. E. Brouwer and A. A. Ivanov (see [2, Proposition 4.3.11]). So our result is a generalization of their result for general s.

On the other hand, we obtained the following result in previous papers [3, 4].

COROLLARY 1.2. ([3, Corollary 1.2] and [4, Corollary 1.4]). *Let Γ be a distance-regular graph with $r = l(1, a_1, b_1)$ and $c_{s+1} = 1$:*
(1) *If $a_1 = 0$, then $s < 2r$, except for either $a_{r+1} = 1$ or $r = 1$.*
(2) *If $a_1 > 0$, then $s < 2r$, except for $r = 1$.*

So the case $r = 1$ was one of our remaining cases.

In Section 4, we investigate subgraphs which are obtained by our theorem to obtain several restrictions of intersection numbers. In particular, we show that there exists no distance-regular graph with $a_1 = 0$, $r = 1$ and $c_4 = 1$.

2. PRELIMINARIES

In this section, we collect several properties which we use in succeeding sections.

Let Γ be a distance-regular graph with $r = l(1, a_1, b_1) = 1$ and $c_{s+1} = 1$ for some positive integer $s \geqslant 2$. Note that $a_1 < a_2$, as $r = 1$ and $c_2 = 1$.

LEMMA 2.1. (1) *There exists no quadruple of vertices (u, v, x, y) such that $\partial_\Gamma(u, y) = j + 1$, $\partial_\Gamma(u, v) = \partial_\Gamma(x, y) = 1$ and $\partial_\Gamma(u, x) = \partial_\Gamma(v, x) = \partial_\Gamma(v, y) = j$, for any $j \leqslant s$.*

(2) *Let α and β be adjacent vertices in Γ. Then $B_j(\alpha, \gamma) = B_j(\beta, \gamma)$ and $C_j(\alpha, \gamma) \cup A_j(\alpha, \gamma) = C_j(\beta, \gamma) \cup A_j(\beta, \gamma)$ for any $j \leqslant s$ and any $\gamma \in \Gamma_j(\alpha) \cap \Gamma_j(\beta)$.*

PROOF. 2.1 (1) Suppose that the statement does not hold. Then we have $x \in C_{j+1}(u, y) - C_j(v, y)$, and this contradicts $c_j = c_{j+1} = 1$.

(2) Take any $\delta \in B_j(\alpha, \gamma)$. From triangle inequalities,

$$j = \partial_\Gamma(\alpha, \delta) - \partial_\Gamma(\alpha, \beta) \leqslant \partial_\Gamma(\beta, \delta) \leqslant \partial_\Gamma(\beta, \gamma) + \partial_\Gamma(\gamma, \delta) = j + 1.$$

Then we have $\partial_\Gamma(\beta, \delta) = j+1$ as otherwise, $(\alpha, \beta, \gamma, \delta)$ contradicts (1). Hence we obtain $B_j(\alpha, \gamma) \subset B_j(\beta, \gamma)$. The rest of the proof is obvious. □

LEMMA 2.2. (1) *Let* $x_0 \sim x_1 \sim x_2$ *be a path of length* 2 *with* $x_0 \not\sim x_2$. *Then there exist* $x_3, x_4 \in \Gamma$ *such that* $(x_0, x_1, x_2, x_3, x_4)$ *is a pentagon.*

(2) *There exists no pentagon* $(x_0, x_1, x_2, x_3, x_4)$ *such that* $x_0, x_1, x_2 \in \Gamma_j(y)$ *and* $x_3, x_4 \in \Gamma_{j+1}(y)$ *for any* $y \in \Gamma$ *and* $j \leqslant s$.

PROOF. (1) Since $a_1 < a_2$, there exists $x_3 \in A_2(x_0, x_2) - A_1(x_1, x_2)$. Let $\{x_4\} = C_2(x_0, x_3)$. It is easy to see that $(x_0, x_1, x_2, x_3, x_4)$ is a pentagon.

(2) We prove the assertion by the induction on j. The case $j = 1$ holds, as otherwise $\{y, x_1\} \subset C_2(x_0, x_2)$, contradicting $c_2 = 1$. Assume that $2 \leqslant j \leqslant s$ and that there exists a pentagon $(x_0, x_1, x_2, x_3, x_4)$ such that $x_0, x_1, x_2 \in \Gamma_j(y)$ and $x_3, x_4 \in \Gamma_{j+1}(y)$. Let $\{z\} = C_j(x_0, y)$. Then we have $\partial_\Gamma(z, x_2) \in \{j-1, j, j+1\}$ from triangle inequality. Suppose that $\partial_\Gamma(z, x_2) = j$. Then we have $x_3 \in B_j(y, x_2) = B_j(z, x_2)$ from Lemma 2.1(2). Thus we have $\partial_\Gamma(z, x_3) = j+1$ and $\partial_\Gamma(z, x_4) = j$, as $\partial_\Gamma(z, x_0) = j-1$. This implies that $\{x_2, x_4\} \subset C_{j+1}(z, x_3)$, contradicting $c_{j+1} = 1$. Suppose that $\partial_\Gamma(z, x_2) = j+1$. Then we obtain $\partial_\Gamma(z, x_1) = j$ as $\partial_\Gamma(z, x_0) = j-1$. So (z, y, x_1, x_2) contradicts Lemma 2.1(1). Hence we have $\partial_\Gamma(z, x_2) = j-1$. Then it is clear that $x_0, x_1, x_2 \in \Gamma_{j-1}(z)$ and $x_3, x_4 \in \Gamma_j(z)$ from $c_j = 1$. This contradicts our inductive assumption. □

LEMMA 2.3. *Let* $y \in \Gamma$ *and* $(x_0, x_1, x_2, x_3, x_4)$ *be a pentagon. Suppose that* $1 \leqslant j \leqslant s$:
(1) *If* $x_0, x_1 \in \Gamma_{j-1}(y)$ *and* $x_2 \in \Gamma_j(y)$, *then* $x_3, x_4 \in \Gamma_j(y)$.
(2) *If* $x_1 \in \Gamma_{j-1}(y)$ *and* $x_0, x_2 \in \Gamma_j(y)$, *then either* $x_3, x_4 \in \Gamma_j(y)$ *or* $x_3, x_4 \in \Gamma_{j+1}(y)$.
(3) *If* $x_0 \in \Gamma_j(y)$, $x_1 \in \Gamma_{j+1}(y)$ *and* $\partial_\Gamma(y, x_3) \leqslant j$, *then* $x_2 \in \Gamma_{j+1}(y)$, $x_3 \in \Gamma_j(y)$ *and* $x_4 \in \Gamma_{j-1}(y)$.

PROOF. The assertions are obvious from $c_j = c_{j+1} = 1$ and Lemma 2.2(2). □

LEMMA 2.4. $a_1 < a_2 < \cdots < a_s$.

PROOF. We prove $a_j < a_{j+1}$ by induction on j. As $a_1 < a_2$, we may assume that $2 \leqslant j \leqslant s-1$. Let $\alpha, z \in \Gamma$ with $\partial_\Gamma(\alpha, z) = j+1$, $\{\beta\} = C_{j+1}(z, \alpha)$ and $\{x\} = C_j(\beta, z)$. From the inductive assumption, there exists $y \in A_j(\alpha, x) - A_{j-1}(\beta, x)$. It is clear that $\partial_\Gamma(\beta, y) = j$. From Lemmas 2.2(1) and 2.3(1), we have $z_3, z_4 \in \Gamma_{j+1}(\alpha)$ such that (y, x, z, z_3, z_4) is a pentagon. As $y \in \Gamma_j(\alpha) \cap \Gamma_j(\beta)$, Lemma 2.1(2) implies that $z_4 \in B_j(\alpha, y) = B_j(\beta, y)$, and hence $z_3 \in \Gamma_{j+1}(\beta)$ from Lemma 2.3(2). This means that $z_3 \in A_{j+1}(\alpha, z) - A_j(\beta, z)$. Since $A_j(\beta, z) \subset A_{j+1}(\alpha, z)$, the assertion follows. □

LEMMA 2.5. *Let* Δ *be an induced subgraph of* Γ. *Suppose that the following three conditions hold for some* $\alpha \in \Delta$ *and for any* $y \in \Gamma_j(\alpha) \cap \Delta$:
(i) $d_\Delta(\alpha) = s$;
(ii) $C_j(\alpha, y) \cup A_j(\alpha, y) \subset \Delta$;
(iii) $B_j(\alpha, y) \cap \Delta \neq \varnothing$ *if* $j \leqslant s-1$.
Then, for any $\beta \in \Delta_1(\alpha)$ *and* $x \in \Gamma_i(\beta) \cap \Delta$, *we have*:
(1) $C_i(\beta, x) \subset \Delta$;
(2) $B_i(\beta, x) \cap \Delta \neq \varnothing$ *if* $i \leqslant s-1$;
(3) $\partial_\Delta(\beta, x) = \partial_\Gamma(\beta, x) = i$.

PROOF. Take any $x \in \Gamma_i(\beta) \cap \Delta$. Let $j := \partial_\Gamma(\alpha, x)$. Then we have $j-1 \leqslant i \leqslant j+1$ from the triangle inequality. Thus we have $c_i = c_j = 1$ as $j \leqslant d_\Delta(\alpha) = s$.

(1), (2) Suppose that $i = j - 1$. Since $c_i = c_{i+1} = 1$ and $a_i < a_{i+1}$, we obtain

$$C_i(\beta, x) = C_{i+1}(\alpha, x) \subset \Delta$$

and

$$\emptyset \neq A_{i+1}(\alpha, x) - A_i(\beta, x) \subset B_i(\beta, x) \cap \Delta$$

from condition (ii). This is the desired result.

Suppose that $i = j$. Lemma 2.1(2) implies that

$$C_i(\beta, x) \subset C_i(\alpha, x) \cup A_i(\alpha, x) \subset \Delta$$

and

$$B_i(\beta, x) \cap \Delta = B_i(\alpha, x) \cap \Delta \neq \emptyset \qquad \text{if } i \leqslant s - 1.$$

The assertion follows.

Suppose that $i = j + 1$. Since $c_i = c_{i-1}$, we have

$$C_i(\beta, x) = C_{i-1}(\alpha, x) \subset \Delta.$$

Now we assume that $i \leqslant s - 1$ and show that $B_i(\beta, x) \cap \Delta \neq \emptyset$. From condition (iii), we obtain $y \in B_{i-1}(\alpha, x) \cap \Delta$. If $y \in B_i(\beta, x)$, then we obtain the desired result. Thus we may assume that $y \in A_i(\beta, x)$ as $C_i(\beta, x) = C_{i-1}(\alpha, x)$. Then we have $z_2 \in B_i(\alpha, y) \cap \Delta = B_i(\beta, y) \cap \Delta$ from condition (iii) and Lemma 2.1(2). Thus there are $z_3, z_4 \in \Gamma$ such that (x, y, z_2, z_3, z_4) is a pentagon. It is clear that $z_3 \in A_{i+1}(\alpha, z_2) \subset \Delta$ and $z_4 \in C_{i+1}(\alpha, z_3) \subset \Delta$ from condition (ii). On the other hand, $z_3, z_4 \in \Gamma_{i+1}(\beta)$ from Lemma 2.3(1). Therefore we obtain $z_4 \in B_i(\beta, x) \cap \Delta$. This is the desired result.

(3) Note that a shortest path connecting β and x in Γ is contained in Δ from (1). The lemma is proved. □

3. Distance-regular Subgraphs

In this section we prove Theorem 1.1. First we define the induced subgraph Δ.

Let u, v be vertices in Γ with $\partial_\Gamma(u, v) = s$. Let Ψ be the subgraph induced by $\Gamma_s(u)$ and let Ψ_v be the connected component of Ψ containing v. Let

$$\Delta := \Delta(u, v) = \bigcup_{x \in \Psi_v} p[u, x].$$

Lemma 3.1. *For any $x \in \Delta$ with $\partial_\Gamma(u, x) = j$, the following hold:*
(1) $\Gamma_i(u) \cap \Delta = \emptyset$ *for any* $i \geqslant s + 1$;
(2) $\partial_\Delta(u, x) = \partial_\Gamma(u, x) = j$;
(3) $d_\Delta(u) = s$;
(4) $C_j(u, x) \subset \Delta$;
(5) $B_j(u, x) \cap \Delta \neq \emptyset$, *if* $j \leqslant s - 1$;
(6) $A_j(u, x) \subset \Delta$.

Proof. (1)–(4) The assertions follow from the definition of Δ.

(5) Since $x \in \Delta$, there exists $z \in \Psi_v$ such that $x \in p[u, z]$. If $j \leqslant s - 1$, then we have

$$\emptyset \neq \Gamma_{j+1}(u) \cap p[u, z] \subset B_j(u, x) \cap \Delta.$$

(6) We prove our assertion by induction on $s - j$. For the case $j = s$, it is clear that $A_s(u, x) \subset \Psi_v \subset \Delta$. We assume that our assertion is true for $j + 1 \leqslant s$ and show that $y \in \Delta$ for any $y \in A_j(u, x)$. Since $j \leqslant s - 1$, we have $w_2 \in B_j(u, x) \cap \Delta$ from (5). Then we

have $w_3, w_4 \in \Gamma_{j+1}(u)$ such that (y, x, w_2, w_3, w_4) is a pentagon, from Lemmas 2.2(1) and 2.3(2). Since $w_2 \in \Delta$, we obtain $w_3 \in A_{j+1}(u, w_2) \subset \Delta$ and $w_4 \in A_{j+1}(u, w_3) \subset \Delta$ from our inductive assumption. Therefore we have $y \in C_{j+1}(u, w_4) \subset \Delta$ from (4). This is the desired result. □

LEMMA 3.2. *Let $w \in \Delta_1(u)$. Then $\Gamma_j(w) \cap \Delta = \emptyset$ for any $j \geqslant s+1$.*

PROOF. Suppose that there exists $x \in \Gamma_j(w) \cap \Delta$ for some $j \geqslant s+1$. Then we have

$$s+1 \leqslant j = \partial_\Gamma(x, w) \leqslant \partial_\Gamma(x, u) + \partial_\Gamma(u, w) \leqslant d_\Delta(u) + 1 = s+1.$$

Thus we have $\partial_\Gamma(w, x) = s+1$ and $\partial_\Gamma(u, x) = s$. Since $w \in \Delta$, there exists $y \in \Psi_v$ such that $w \in p[u, y]$. Note that $y \in \Gamma_{s-1}(w) \cap \Psi_v$, $x \in \Gamma_{s+1}(w) \cap \Psi_v$ and Ψ_v is connected. Thus there exists an edge $\gamma \sim \delta$ such that $\gamma \in \Gamma_s(w) \cap \Psi_v$ and $\delta \in \Gamma_{s+1}(w) \cap \Psi_v$. Then (w, u, γ, δ) contradicts Lemma 2.1(1). We have our assertion. □

LEMMA 3.3. *For any $z \in \Delta$ and $x \in \Gamma_j(z) \cap \Delta$, the following hold*:

(1) $\Gamma_i(z) \cap \Delta = \emptyset$ *for any* $i \geqslant s+1$;
(2) $\partial_\Delta(z, x) = \partial_\Gamma(z, x) = j$;
(3) $B_j(z, x) \cap \Delta \neq \emptyset$, *if* $j \leqslant s-1$;
(4) $C_j(z, x) \subset \Delta$;
(5) $A_j(z, x) \subset \Delta$;
(6) $d_\Delta(z) = s$.

PROOF. We prove the assertions by induction on $t = \partial_\Gamma(u, z)$. For the case $t = 0$, the assertions follow from Lemma 3.1. Let $t \geqslant 1$ and $\{w\} = C_t(u, z)$

(1) We may assume that $t \geqslant 2$ from Lemma 3.2. Thus let $\{y\} = C_{t-1}(u, w)$ and $x, x' \in \Gamma$ such that (y, w, z, x, x') is a pentagon. Then it is clear that $x \in A_t(u, z) \subset \Delta$ and $x' \in C_t(u, x) \subset \Delta$ from Lemma 3.1. Suppose that there exists $\gamma \in \Gamma_j(z) \cap \Delta$ for $j \geqslant s+1$ to derive a contradiction. From the inductive assumption and the triangle inequality we have

$$s+1 \leqslant j = \partial_\Gamma(\gamma, z) \leqslant \partial_\Gamma(\gamma, w) + \partial_\Gamma(w, z) \leqslant d_\Delta(w) + 1 = s+1.$$

Hence we obtain $\partial_\Gamma(\gamma, z) = s+1$ and $\partial_\Gamma(\gamma, w) = s$. Since $\partial_\Gamma(\gamma, x') \leqslant d_\Delta(x') = s$, Lemma 2.3(3) implies that $\partial_\Gamma(y, \gamma) = s-1$. Take any $\delta \in A_s(w, \gamma) \subset A_{s+1}(z, \gamma)$. Since $\delta \in A_s(w, \gamma) \subset \Delta$, we obtain $\partial_\Gamma(\delta, y) = s-1$ from Lemma 2.3(3) as above. This means that $A_s(w, \gamma) \subset A_{s-1}(y, \gamma)$, contradicting Lemma 2.4.

(2)–(4) The assertions follow from the inductive assumption and Lemma 2.5.

(5) We prove by induction on $s-j$. First we show that $A_s(z, x) \subset \Delta$ for any $x \in \Gamma_s(z) \cap \Delta$. From the triangle inequality and $d_\Delta(w) = s$, we have $\partial_\Gamma(w, x) \in \{s-1, s\}$. If $\partial_\Gamma(w, x) = s$, then Lemma 2.1 means that $A_s(z, x) \subset C_s(w, x) \cup A_s(w, x) \subset \Delta$ from the inductive assumption. Suppose that $\partial_\Gamma(w, x) = s-1$. Take any $y \in A_s(z, x)$. We may assume that $y \in B_{s-1}(w, x)$, as otherwise we obtain $y \in A_{s-1}(w, x) \subset \Delta$. By (3), there exists $x_2 \in B_{s-1}(w, x) \cap \Delta$. Then $\partial_\Gamma(z, x_2) = s$ from (1). If $\partial_\Gamma(y, x_2) \leqslant 1$, then $y \in A_s(w, x_2) \cup \{x_2\} \subset \Delta$. Thus we assume that $\partial_\Gamma(y, x_2) = 2$ and hence that there exist $x_3, x_4 \in \Gamma$ such that (y, x, x_2, x_3, x_4) is a pentagon. From Lemma 2.3(3), we have either $x_3, x_4 \in \Gamma_s(w)$ or $x_3, x_4 \in \Gamma_{s+1}(w)$. If $x_3, x_4 \in \Gamma_{s+1}(w)$, then $x_3 \in B_s(w, x_2) = B_s(z, x_2)$ and $x_4 \in B_s(w, y) = B_s(z, y)$, by Lemma 2.1(2). This implies that $y, x, x_2 \in \Gamma_s(z)$ and $x_3, x_4 \in \Gamma_{s+1}(z)$, contradicting Lemma 2.2(2). Thus we have $x_3, x_4 \in \Gamma_s(w)$. Then $x_3 \in A_s(w, x_2) \subset \Delta$, $x_4 \in A_s(w, x_3) \subset \Delta$ and $y \in A_s(w, x_4) \subset \Delta$. Hence we obtain $A_s(z, x) \subset \Delta$. The rest of the proof is the same as that of Lemma 3.1(6).

(6) This is a direct consequence of (1), (2) and (3). □

PROOF OF THEOREM 1.1. Take any $x \in \Delta$. We have $d_\Delta(x) = s$ from Lemma 3.3(6). Thus there exists $z \in \Delta_s(x)$. Note that $x \in \Delta_s(z)$ and $\Gamma_{s+1}(z) \cap \Delta = \varnothing$. We have

$$\Delta_1(x) = A_s(z, x) \cup C_s(z, x)$$

by Lemma 3.3(4), (5). Hence

$$k_\Delta = |\Delta_1(x)| = a_s(\Gamma) + c_s(\Gamma)$$

does not depend on the choice of x. Furthermore, we obtain that

$$c_j(\Delta) = c_j(\Gamma) = 1, \qquad a_j(\Delta) = a_j(\Gamma) \qquad \text{and} \qquad b_j(\Delta) = k_\Delta - c_j(\Delta) - a_j(\Delta)$$

depend only on j. Therefore Δ is distance-regular. The theorem is proved. □

4. COROLLARIES OF THE THEOREM

In this section, we give several corollaries obtained from our theorem. First we introduce the following result, which has been proved by A. E. Brouwer and A. A. Ivanov.

PROPOSITION 4.1 ([2, §4.3 C]). *Let Γ be a distance-regular graph with $|\Gamma| = n$, valency k, $c_3 = 1$ and $l(1, a_1, b_1) = 1$. Then the following hold*:
(1) $(a_2 - a_1) \mid b_1$;
(2) $(a_2 + 1)(a_2 - a_1) \mid kb_1$;
(3) $(a_2 + 1)(a_2 - a_1)\{1 + (a_2 + 1)(a_2 - a_1 + 1)\} \mid nkb_1$.
Moreover, if $0 = a_1 < a_2$, *then* $a_2 \in \{1, 2, 6, 56\}$.

This result is obtained by counting the number of subgraphs Δ of diameter 2.

Let Γ be a distance-regular graph with $c_{s+1} = 1$ for some positive integer $s \geq 2$. Then, from Theorem 1.1, we have a sequence of subgraphs

$$\{u\} = \Delta^0 \subset \Delta^1 \subset \Delta^2 \subset \Delta^3 \subset \cdots \subset \Delta^s \subset \Gamma,$$

where Δ^j is a subgraph of diameter j. Hence we obtain several results by counting the numbers of subgraphs.

PROPOSITION 4.2. *Let Γ be a distance-regular graph with $|\Gamma| = n$, valency k, $c_{s+1} = 1$ and $l(1, a_1, b_1) = 1$. If $2 \leq m \leq s$, then the following hold*:
(1) $(a_m - a_{m-1}) \mid b_{m-1}$;
(2) $(a_m + 1) \prod_{j=1}^{m-1} (a_m - a_j) \mid k \prod_{j=1}^{m-1} b_j$;
(3) $\delta_m(a_m + 1) \prod_{j=1}^{m-1} (a_m - a_j) \mid nk \prod_{j=1}^{m-1} b_j$;
where $\delta_m := |\Delta^m| = 1 + (a_m + 1) + (a_m + 1) \sum_{j=2}^{m} \{\prod_{i=1}^{j-1} (a_m - a_i)\}$.

PROOF. We count the number of subgraphs Δ^m in the graph Γ, on a given pair of vertices at distance $m - 1$, on given vertices, and their total number, respectively. □

PROPOSITION 4.3. *Let Γ be a distance-regular graph with valency k, $c_{s+1} = 1$ and $l(1, a_1, b_1) = 1$. If $2 \leq h < m \leq s$, then the following hold*:
(1) $(a_h - a_{h-1}) \mid (a_m - a_h)$;
(2) $(a_h + 1) \prod_{j=1}^{h-1} (a_h - a_j) \mid (a_m + 1)\prod_{j=1}^{h-1} (a_m - a_j)$;
(3) $\delta_h(a_h + 1)\prod_{j=1}^{h-1} (a_h - a_j) \mid \delta_m(a_m + 1)\prod_{j=1}^{h-1} (a_m - a_j)$;
where $\delta_i := |\Delta^i|$.

PROOF. We count the number of subgraphs Δ^h in the graph Δ^m, on given pair of vertices at distance $h-1$, on given vertices, and their total number, respectively. □

Next we prove the following result.

THEOREM 4.4. *Let Γ be a distance-regular graph with $d=3$ and $c_3=1$. If a_2 is fixed, then there exist only finitely many such distance-regular graphs.*

To prove the theorem, we need some preliminaries. For the case $a_1=a_2$, it is well known that the 7-gon is the only such graph from the classification of Moore graphs and Moore geometries (see [2, pp. 205–208]). Hence we may assume that $a_1<a_2$.

Let $A:=A(\Gamma)$ be the adjacency matrix of Γ. Let $k=\theta_0>\theta_1>\theta_2>\theta_3$ be the eigenvalues of A, which are called the eigenvalues of Γ, and let m_j denote the multiplicity of θ_j in A.

PROPOSITION 4.5. *The following hold:*
(1) θ_1, θ_2 and θ_3 *are roots of the equality*

$$F_3(x)=x^3+(1-a_1-a_2)x^2+(2-2k+a_2a_1)x+(a_2k+1-k)=0.$$

(2)
$$m_j=\frac{|\Gamma|\,kb_1b_2}{(k-\theta_j)F_3'(\theta_j)F_2(\theta_j)}.$$

where $F_2(x)=x^2+(1-a_1)x+(1-k)$.
(3) *$F_3(x)$ is reducible on the rational field $\mathbf{Q}$. In particular, Γ has at least one integral eigenvalue.*
(4) *Let θ be an integral eigenvalue of Γ. Then*

$$(2\theta-a_2+1)k=\theta^3-(a_2-1+a_1)\theta^2+(a_1a_2+2)\theta+1.$$

In particular, $2\theta\neq a_2-1$.

PROOF. (1), (2) These are well known basic properties. See [1, Chapter III].

(3) Suppose the contrary. Then θ_1, θ_2 and θ_3 are algebraic conjugates over $\mathbf{Q}$. Then we have $m=m_1=m_2=m_3$. Note that $\theta_1+\theta_2+\theta_3=(a_1+a_2-1)\geqslant 0$. We have

$$0=\mathrm{tr}(A)=\sum_{j=0}^{3}\theta_jm_j=k+(\theta_1+\theta_2+\theta_3)m\geqslant k.$$

This is a contradiction.

(4) The first part follows from (1). Suppose that $2\theta=a_2-1$. Then we obtain

$$\begin{aligned}0&=\theta^3-(2\theta+a_1)\theta^2+(2a_1\theta+a_1+2)\theta+1\\&=-(\theta+1)\{\theta^2-(a_1+1)\theta-1\}.\end{aligned}$$

This is impossible, as θ and a_1 are non-negative integers. □

PROOF OF THEOREM 4.4. From Proposition 4.5(4), we have

$$8k=P(\theta,a_2,a_1)-\frac{R(a_2,a_1)}{(2\theta-a_2+1)}.$$

where

$$P(\theta,a_2,a_1)=4\theta^2+(2-2a_2-4a_1)\theta+2a_2+2a_1+7+2a_2a_1-a_2^2$$

and

$$R(a_2, a_1) = (a_2 + 1)\{a_2^2 - (2a_1 + 4)a_2 + 2a_1 - 1\}.$$

Suppose that a_2 is fixed. As $a_1 < a_2$, there are only finitely many possible values for a_1. We fix a_1. Then $R(a_2, a_1)$ is determined. We remark that $R(a_2, a_1) \neq 0$ for any $0 \leq a_1 < a_2$. Hence $(2\theta - a_2 + 1)$ must be a divisor of $R(a_2, a_1)$. Thus there are only finitely many possible values for θ, and hence there are only finitely many possible values for k. This proves the theorem. □

Corollary 4.6. *There are no distance-regular graphs with $d = 3$, $0 = a_1 < a_2$ and $c_3 = 1$.*

Proof. Suppose the contrary. From Proposition 4.1, we have that $a_2 \in \{1, 2, 6, 56\}$. Thus there are only a finite number of feasible parameters from Theorem 4.4, and all of them can be ruled out by Proposition 4.1 and the integrality condition of m_j from Proposition 4.5(2). □

Corollary 4.7. *Let Γ be a distance-regular graph with $a_1 = 0$ and $r = 1$. Then $c_4 \neq 1$.*

Proof. This is a direct consequence of Theorem 1.1 and Corollary 4.6. □

Acknowledgements

The author would like to thank Professor Hiroshi Suzuki for his valuable suggestions and comments. This work was supported in part by a grant from the Japan Society for the Promotion of Science.

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Received 4 May 1995 and accepted in revised form 23 October 1995

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